USE OF TRIANGULAR DISTRIBUTION TO SOLVE SINGLE OBJECTIVE ASSIGNMENT PROBLEMS UNDER UNCERTAINTIES

C.P.S. Pathirana[1*] and W.B. Daundasekera[2]

[1]Department of Mathematics and Philosophy of Engineering,
The Open University of Sri Lanka, Sri Lanka
[2]Department of Mathematics, University of Peradeniya, Sri Lanka

This study investigates the use of Triangular distribution in solving single objective Assignment problems under uncertainties. The Assignment Problem (AP) is one of the fundamental problems in the area of combinatorial optimization. The classical AP assumes that the assignment costs (AC) corresponding to the assignments are deterministic. But, in many real world problems, AC are far from deterministic. In fact, it can be observed that AC behave in an unpredictable random manner. Also, it can be observed that the AC cannot be estimated precisely due to insufficient information related to the application. This study assumes that an assignment cost can fluctuate between two estimated values known as optimistic and pessimistic values in a probabilistic manner, where most probable value lies in between. Therefore, in this study, it is assumed that these random cost coefficients follow the Triangular distribution. This has been proven to be a realistic assumption. Applying the Monte Carlo simulation technique, random AC were estimated. Subsequently, the AP was converted to a deterministic model and hence, solved it using the Hungarian algorithm. To illustrate this approach, a randomly generated AP was formed based on the Triangular distribution. Subsequently, the Monte Carlo simulation is conducted to the AP, where each time the optimum solution is recorded. Based on the results obtained using this technique, a statistical analysis is performed to understand the sensitivity of the optimum solution of the uncertain model. This in-depth analysis concludes that this technique provides more realistic answers to the optimal assignment. This process was done with the help of a python program. This study encourages researchers to test other suitable probability distributions according to the nature of assignment costs of the AP.

Keywords: assignment problem, uncertainty, Triangular distribution, Monte Carlo simulation, Hungarian algorithm.

[*]Corresponding Author: cppat@ou.ac.lk

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C.P.S. Pathirana[1*] and W.B. Daundasekera[2]
[1]Department of Mathematics and Philosophy of Engineering,
The Open University of Sri Lanka, Sri Lanka
[2]Department of Mathematics, University of Peradeniya, Sri Lanka

INTRODUCTION

The Assignment Problem (AP) is one of the fundamental problems in the area of combinatorial optimization. Easter- field (Easterfield, 1946) first studied the algorithm for the classic assignment problem. The classical AP assumes that the assignment costs (AC) corresponding to the assignments are deterministic. But, in many real-world problems, AC are far from deterministic.

In real world applications, some uncertain factors can appear because of the lack of historical data, insufficient information, or some other uncontrollable situations in determining the AC. The regular existing algorithms fail to solve the uncertain AP. Furthermore, researchers employed probability theory to deal with these uncertain factors (Kakran, 2022). Some researchers started to apply the uncertainty theory in order to study various uncertain problems (Bo Zhang et al. 2013).

This study assumes that a AC can fluctuate between two estimated values known as optimistic and pessimistic values in a probabilistic manner, where the most probable value lies in between. Therefore, in this study, it is assumed that these AC follow the Triangular distribution. This has been proven to be a realistic assumption. Applying the Monte Carlo simulation technique, AC are estimated. Subsequently, the AP was converted to a deterministic model and hence, solved it using the Hungarian algorithm.

METHODOLOGY

The AP under the uncertainty can be generally given as follows:

$$Min\ Z = \sum_{i=1}^{n} \sum_{j=1}^{n} \zeta_{ij} x_{ij}$$

subject to the constraints:

$$\sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, 2, 3, \dots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, 2, 3, \dots, n$$

$$x_{ij} = 0\ or\ 1, \quad i, j = 1, 2, 3, \dots, n$$

where x_{ij} is defined as

$$x_{ij} = \begin{cases} 1, & if\ worker\ i\ is\ assigned\ to\ job\ j, \\ 0, & otherwise. \end{cases}$$

Here, ζ_{ij} is the assignment cost corresponding to the i^{th}worker assigned to j^{th} job. In our study this is considered to be a random variable which follows a Triangular distribution. The Triangular distribution has three parameters: the optimistic (mimimum) value a, the pessimistic (maximum) value b, and the most likely value c. That is,

$$\zeta_{ij} \sim Tr(a, c, b). \qquad \text{where } a \le c. \le b.$$

A Triangular distribution is a continuous distribution with a probability density function has a triangular shape in appearance. The Triangular commutative distribution can be defined as:

$$F(x) = \begin{cases} \dfrac{(x-a)^2}{(b-a)(c-a)}, & a \le x < c \\ 1 - \dfrac{(b-x)^2}{(b-a)(b-c)}, & c \le x \le b. \end{cases}$$

Thus, the inverse cumulative distribution function can be derived as:

$$f^{-1}(u) = \begin{cases} a + \sqrt{(b-a)(c-a)u}, & 0 < u < \dfrac{(c-a)}{(b-a)} \\ b - \sqrt{(b-a)(b-c)(1-u)}, & \dfrac{(c-a)}{(b-a)} \le u < 1. \end{cases}$$

where 0< u <1.

This proposes the following three-step process to obtain optimal assignment for an AP with uncertain cost coefficients:

Step 1: Generate different values of $\zeta_{ij} \sim Tr(a, c, b)$ by varying the value of *u* between 0 and 1.

Step 2: Using the values of ζ_{ij} generated in Step 1, estimate the cost coefficients.

Step 3: Using the estimated cost coefficients calculated in Step 2, apply Hungarian algorithm to obtain the optimal assignment.

RESULTS AND DISCUSSION

To illustrate the method, consider the following assignment model.

$$Min\ Z = \sum_{i=1}^{5} \sum_{j=1}^{5} \zeta_{ij} x_{ij}$$

Here, there are 5 workers to be assigned optimally to 5 jobs, where assignment costs ζ_{ij} are randomly generated according to the Triangular distribution and complete cost coefficient matrix is given below:

$$\zeta = \begin{pmatrix} (0.02, 5.49, 7.83) & (0.32, 5.08, 7.93) & (0.84, 4.02, 11.00) & (1.70, 5.73, 9.67) & (1.18, 6.36, 7.10) \\ (2.61, 4.66, 10.52) & (2.17, 6.39, 8.56) & (0.54, 4.97, 9.57) & (2.77, 4.39, 7.32) & (1.45, 6.78, 9.53) \\ (0.14, 4.76, 8.70) & (3.83, 4.47, 10.91) & (1.21, 6.89, 9.83) & (0.31, 5.40, 9.68) & (1.94, 6.75, 8.22) \\ (1.19, 4.67, 7.76) & (0.27, 6.46, 9.55) & (3.82, 5.47, 8.17) & (3.87, 6.86, 9.40) & (3.25, 5.12, 7.38) \\ (3.59, 4.68, 8.05) & (1.50, 6.41, 10.61) & (3.32, 5.03, 10.88) & (1.10, 6.39, 9.56) & (0.29, 6.01, 8.14) \end{pmatrix}$$

u values are randomly generated and for each u value AC are estimated. To illustrate the procedure, the following table exhibits the estimated AC for u = 0.025.

Table 1. Assignment Costs for u = 0.025

ζ_{ij}	$f^{-1}(u)$	ζ_{ij}	$f^{-1}(u)$	ζ_{ij}	$f^{-1}(u)$	ζ_{ij}	$f^{-1}(u)$	ζ_{ij}	$f^{-1}(u)$
ζ_{11}	1.0534	ζ_{21}	3.2467	ζ_{31}	1.1343	ζ_{41}	1.9460	ζ_{51}	3.9386
ζ_{12}	1.2716	ζ_{22}	2.9911	ζ_{32}	4.1666	ζ_{42}	1.4684	ζ_{52}	2.5575
ζ_{13}	1.7387	ζ_{23}	1.5400	ζ_{33}	2.3164	ζ_{43}	4.2436	ζ_{53}	3.8885
ζ_{14}	2.5961	ζ_{24}	3.1993	ζ_{34}	1.4019	ζ_{44}	4.5129	ζ_{54}	2.1577
ζ_{15}	2.0556	ζ_{25}	2.4876	ζ_{35}	2.8090	ζ_{45}	3.6894	ζ_{55}	1.3495

Subsequently, Monte Carlo simulation was performed to the AP, where in each replication the optimum solution to the AP is recorded. Based on the results obtained in the above technique, a statistical analysis is performed to understand the sensitivity of the optimum solution. Subsequently, optimum solution is obtained by using the inverse cumulative Triangular distribution by varying *u* value. According to Step 3 mentioned above, the optimal assignments and assignment values of all replicates for each *u* value are obtained. The simulated data and results are shown in the Table 2 below:

Table 2. List of optimal assignments

u	*Optimal Assignment*	*Optimum value*	u	*Optimal Assignment*	*Optimum value*
0.025	{(1,1), (2, 3), (3,4), (4, 2), (5, 5)}	6.8132	0.525	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	24.9227
0.050	{(1,1), (2, 3), (3,4), (4, 2), (5, 5)}	9.0431	0.550	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	25.4590
0.075	{(1,1), (2, 3), (3,4), (4, 2), (5, 5)}	10.7541	0.575	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	25.9976
0.100	{(1,1), (2, 3), (3,4), (4, 2), (5, 5)}	12.1966	0.600	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	26.5028
0.125	{(1,1), (2, 3), (3,4), (4, 2), (5, 5)}	13.4674	0.625	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	26.9648
0.150	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	14.5524	0.650	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	27.4339
0.175	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	15.5061	0.675	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	27.9144
0.200	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	16.3937	0.700	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	28.4077
0.225	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	17.2274	0.725	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	28.9160
0.250	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	18.0159	0.750	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	29.4440
0.275	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	18.7659	0.775	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	29.9990
0.300	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	19.4826	0.800	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	30.5859
0.325	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	20.1699	0.825	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	31.2107
0.350	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	20.8312	0.850	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	31.3358
0.375	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	21.4694	0.875	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	32.6120
0.400	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	22.0866	0.900	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	33.4194
0.425	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	22.6848	0.925	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	34.3358
0.450	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	23.2656	0.950	{(1,2), (2, 4), (3,1), (4, 3), (5, 5)}	36.7411
0.475	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	23.8305	0.975	{(1,2), (2, 4), (3,5), (4, 3), (5, 1)}	36.7411
0.500	{(1,2), (2, 3), (3,4), (4, 1), (5, 5)}	24.3811			

Optimum solution against *u* is given in the Figure 1 below:

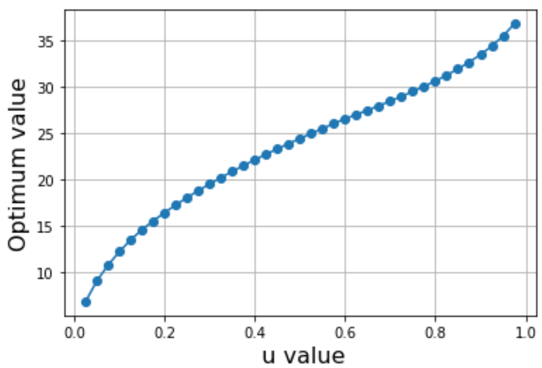

Figure 1. Optimum solution against *u*

CONCLUSIONS AND RECOMMENDATIONS

In this study an AP problem with uncertain assignment costs was considered. It is assumed that the assignment costs follow Triangular distribution. When the assignment costs are uncertain, finding the optimum assignment satisfying the minimum total assignment cost is challenging. In this study we use Monte Carlo simulation to estimate the assignment costs and subsequently transformed the problem to a deterministic assignment problem. Hence, the optimum total assignment cost and the optimum assignments are determined using the Hungarian algorithm. The method is illustrated using a randomly generated AP. For the considered AP it was found that the optimum assignment cost is 6.8 when u = 0.025. Also, the optimum assignment is {(1, 1), (2, 3), (3, 4), (4, 2), (5, 5)} and the total assignment uncertain cost is calculated as:

$$\sum_{i=1}^{5}\sum_{j=1}^{5}\xi_{ij}x_{ij} = 1.0534 + 1.5400 + 1.4019 + 1.4684 + 1.3495 = 6.8132$$

This study has investigated an AP with uncertain AC. It is assumed that the AC follow a Triangular distribution. But, in certain applications the distribution could vary accordingly. This study encourages to explore AC which follow other continuous distributions.

REFERENCES

Bo Zhang, Peng, J., (2013). Uncertain programming model uncertain optimal assignment problem. Applied Mathematical Modelling 37, 6458–6468.

Easterfield, T.E.(1946). A combinatorial algorithm, J. Lond. Math. Soc. 21(3), 219-226.

Kakran, V. Y.,(2022). PhD thesis, S.V. National Institute of Technology, Surat, Gujarat, India.